YDK 517.9

BOUNDARY VALUE PROBLEMS CONNECTED WITH A POTENTIAL FLOW IN A DOUBLE-CONNECTED DOMAIN

©Leonid R. Volevich

The lecture is devoted to the boundary value problems arising in mathematical analysis of a plane subsonic flow in a simply or double-connected domain with every component of the boundary having the leading edge. The Newton's linearization and the conformal mapping of the domain permit to reduce the original physical problem to a sequence of boundary value problems in a rectangular domain. The latter problems can be inverstigated numerically using finite elements of high order (bicubic elements). But in the present lecture we shall not dwell on the computational side (see [1-5]) concentrating on the mathematical side of the problem. There is an extensive literature devoted to potential flows. A lot of referenses in this literature can be found by the reader in [6].

§1. Mathematical formulation of a potential flow problem.

1.1 Full potential equation. We shall treat non-stationary, inviscid, irrotational adiabatic and isentropic flow of an ideal gas in $\mathbb{R}^3$. Such a flow is characterized by conservation of the mass equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0,$$

the momentum equation

$$\frac{\partial v}{\partial t} + (v, \nabla) + \frac{1}{\rho}\nabla P = 0;$$

and the adiabatic isentropic ideal gas law: $P = \mathrm{const}\rho^k$. Here, as usual, ρ, P and v are correspondingly the density, pressure and the velocity of the flow. If the flow is irrotational, i.e. rot $v = 0$, then the velocity possesses a potential, $v = \nabla\varphi$, and conservation of mass equation may be rewritten in the form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\nabla\varphi) = 0. \tag{1.1}$$

If we substitute $v = \nabla\varphi$ in the momentum conservation equation and suppose that the flow is stable at infinity we obtain (after integration) the Cauchy–Lagrange integral:

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 + \frac{a^2}{k-1}\left(\frac{\rho}{\rho_\infty}\right)^{k-1} = \frac{1}{2}|v_\infty|^2 + \frac{a_\infty^2}{k-1}, \tag{1.2}$$

where ρ_∞, v_∞ and $M_\infty = |v_\infty|/a_\infty$ are, respectively, values at infinity of ρ, v, a (sound velocity) and M (Mach number). Without loss generality we can suppose that $\rho_\infty = |v_\infty| = 1$ (in the opposite case we can change the scales). Then solving equation (2) for ρ we obtain

$$\rho = \left(1 + \frac{k-1}{2}M_\infty^2(1 - |\nabla\varphi|^2) - (k-1)M_\infty^2\varphi_t\right)^{\frac{1}{(k-1)}}. \tag{1.3}$$

Differentiating this equation with respect to t and substituting the result in equation (1) we come to non-stationary potential equation

$$M_\infty^2\rho^{2-k}\left(\varphi_{tt} + (\nabla\varphi,\ \nabla\varphi_t)\right) - \operatorname{div}(\rho\nabla\varphi) = 0. \tag{1.4}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1.2 Initial and boundary value problem. Let G be a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂G and ν is the internal normal to ∂G. In $\mathbb{R}^3 \setminus G$ we seek a solution $\varphi(x,t)$ of equation (4) satisfying the boundary condition

$$\left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial G} = 0. \tag{1.5}$$

and the condition at infinity

$$(\nabla\varphi)(x,t) \to v_\infty, \quad |x| \to \infty, \tag{1.6}$$

where v_∞ is the velocity of the onset flow. For $t = 0$ we pose the initial conditions

$$\left.\varphi\right|_{t=0} = \varphi_0(x), \ \left.\varphi_t\right|_{t=0} = \varphi_1(x). \tag{1.7}$$

Let $\varphi_\infty(x) = \langle v_\infty, x\rangle$ be the potential of the onset flow. We pose

$$\varphi(x,t) = \varphi_\infty(x) - \Phi(x,t).$$

Then $\Phi(x,t)$ is the solution of the problem

$$M_\infty \rho^{2-k}(\Phi_{tt} - \langle\nabla(\Phi - \varphi_\infty), \nabla\Phi_t\rangle) - \mathrm{div}(\rho\nabla(\Phi - \varphi_\infty)) = 0,$$

$$\left.\frac{\partial\Phi}{\partial\nu}\right|_{\partial G} = \langle v_\infty, \nu\rangle, \ |\nabla\Phi(x,t)| \to 0, \ |x| \to \infty.$$

We can assume, that the function $\Phi(x,t)$ for every fixed t is an element of Sobolev's space H^1: $\Phi(x,t) \in H^1(\mathbb{R}^3 \setminus G)$, $\forall t > 0$.

1.3 Semi-integral form of the problem (1.4)–(1.7). When we construct a discrete approximation of the problem under investigation we usually deal with the integral form of the problem. We can state the following

Proposition. The smooth function $\Phi(x,t)$, Φ, Φ_{x_1}, Φ_{x_2}, Φ_{x_3} $\in$ $L_2(\mathbb{R}^3 \setminus G)$, is a solution of (1.4′)-(1.6′) if and only if the integral identity

$$\iiint_{\mathbb{R}^3\setminus G} [M_\infty^2 \rho^{2-k}(\Phi_{tt} - \langle\nabla(\Phi - \Phi_\infty), \nabla\Phi_t\rangle)\omega +$$
$$\rho\langle\nabla(\Phi - \varphi_\infty), \nabla\omega\rangle + \langle\nabla\varphi_\infty, \nabla\omega\rangle]dx - \iint_{\partial G} \langle v_\infty, \nu\rangle\omega ds = 0 \tag{1.8}$$

holds for every test function $\omega(x)$. The proposition reduces the boundary and initial problem for equation (1.4) to the integral equation (1.8) with the initial condition (1.7).

1.4 The stationary problem in the integral form and the variational principle. If the potential of the flow does not depend on time, the problem (1.7), (1.8) transforms into the steady problem

$$\iiint_{\mathbb{R}^3\setminus G} \langle\rho\nabla(\Phi - \varphi_\infty) + \nabla\varphi_\infty, \nabla\omega\rangle dx - \iint_{\partial G} \langle v_\infty, \nu\rangle\omega ds = 0, \ \forall\omega \tag{1.9}$$

where

$$\rho = (1 + \frac{k-1}{2} M_\infty^2 (1 - |\nabla(\Phi - \varphi_\infty)|^2)^{\frac{1}{k-1}}. \tag{1.10}$$

It is possible to obtain equation (1.9) by means of the variational principle formulated by Bateman in the case of interior flows and by M.Shiffman [7] in the plane case. The stationary points of the functional

$$\mathcal{F}(\Phi)=\iiint\limits_{\mathbb{R}^3\setminus G}\left[F(|\nabla(\Phi-\varphi_\infty)|-F(1)+\langle\nabla\varphi_\infty,\nabla\Phi\rangle\right]dx-\iint\limits_{\partial G}\langle\varphi_\infty,\nu\rangle\Phi ds,$$

where $F(q)=\int_0^q s\rho(s)ds$, $\rho(q)=\left(1+\frac{k-1}{2}M_\infty^2(1-q^2)\right)^{\frac{1}{k-1}}$, are the solutions of relation (1.9).

Indeed, if Φ is a stationary point and ω is an arbitrary test function, then

$$\frac{d}{d\varepsilon}\mathcal{F}(\Phi+\varepsilon\omega)\Big|_{\varepsilon=0}=0. \tag{1.11}$$

Taking note of the equality

$$\frac{d}{d\varepsilon}\mathcal{F}(|\nabla(\Phi+\varepsilon\omega-\varphi_\infty)|)\Big|_{\varepsilon=0}=\rho(|\nabla(\Phi-\varphi_\infty)|)\langle\nabla(\Phi-\varphi_\infty),\nabla\omega\rangle$$

we obtain that (1.11) is equivalent to (1.9).

1.5 Linearization of the problem (1.9). Newton's method.

Denote by $F(\Phi,\omega)$ the left-hand side of (1.9). Then to solve the equation

$$F(\Phi,\omega)=0,\ ,\ \forall\omega\in H^1(\mathbb{R}^3\setminus G) \tag{1.12}$$

we use Newton's method. Suppose that $\Phi=\Phi^{(n)}$ is the n-th iteration of the solution and $\Phi^{(n+1)}=\Phi+\Psi$. According to Newton's method the correction Ψ is a solution of the linear equation

$$F(\Phi,\Psi,\omega)=-F(\Phi,\omega), \tag{1.13}$$

where

$$F(\Phi,\Psi,\omega)=\frac{d}{d\varepsilon}F(\Phi+\varepsilon\Psi,\omega)\big|_{\varepsilon=0}=$$
$$\iiint\limits_{\mathbb{R}^3\setminus G}\int\left[\rho(\Phi)\langle\nabla\Psi,\nabla\omega\rangle-M_\infty^2\rho^{2-k}(\Phi)\langle\nabla(\Phi-\varphi_\infty),\nabla\omega\rangle\langle\nabla\Psi,\nabla\omega\rangle\right]dx. \tag{1.14}$$

In other words, every step of Newton's iterations is reduced to solving a linear second order equation

$$\operatorname{div}(\rho\nabla\Psi)+\ldots(\text{lower terms})=f$$

with zero Neumann condition on the boundary ∂G. As for the first iteration, in this case $f\equiv 0$, and Neumann's boundary condition is nonzero.

§2.Flow past a profile with an edge.

2.1 Formulation of the problem. Now we shall consider two dimensional flows in the plane $\mathbb{R}^2$ with the coordinates x and y. Suppose that $G\in\mathbb{R}^2$ is a bounded domain (wing section) with a boundary ∂G which is a smooth curve up to some point $z_0=(x_0,y_0)$ (trailing edge) where the tangents to the curve form an acute angle.

Let $\varphi(x,y)$ be the potential of the flow in the exterior of G. Then it satisfies the potential equation

$$\frac{\partial}{\partial x}(\rho\frac{\partial\varphi}{\partial x})+\frac{\partial}{\partial y}(\rho\frac{\partial\varphi}{\partial y})=0, \tag{2.1}$$

nonpenetration condition on the boundary

$$\left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial G\setminus z_0} = 0 \tag{2.2}$$

and the Kutta-Zhukovsii condition on the edge

$$|\nabla\varphi(x,y)| < \infty, \quad (x,y)\in\overline{\mathbb{R}^2\setminus G}. \tag{2.3}$$

The conditions (2.1), (2.2) and (2.3) must be complemented by the condition at infinity:

$$\nabla\varphi(x,y)\to v_\infty = (\cos\alpha,\sin\alpha) \text{ as } |x|+|y|\to\infty, \tag{2.4}$$

where α is the angle of attack.

As a rule, the function φ satisfying the above condition is not univalent, i.e.

$$\gamma = \frac{1}{2\pi}\int\limits_C d\varphi \neq 0, \tag{2.5}$$

where C is a closed curve in $\mathbb{R}^2\setminus G$ enveloping ∂G. The number γ from (2.5) is circulation; it is an additional parameter to satisfy the Kutta-Zhukovskii condition.

2.2 Transformation of the domain of the flow. According to whell- known Riemann's theorem there exists a conformal map $z = F(\omega)$ of the unit disk $\{\omega\in\mathbb{C}, |\omega|\leqslant 1\}$ onto the exterior of G in the plane $z = x+iy$. This map transforms the circle $\{\omega\in\mathbb{C},\ |\omega| = 1\}$ into the boundary ∂G, and we can suppose that the trailing edge z_0 is the image of the point $e^{\pi i}$: $F(e^{\pi i}) = z_0$. On the other hand, the function $\omega = -e^{i\zeta}$, $\zeta = \xi + i\eta$ transforms the half-strip $\Pi = \{\zeta\in\mathbb{C}, 0\leqslant\xi\leqslant 2\pi, \eta\geqslant 0\}$ in the ζ-plane into the unit disk in the ω-plane, the unit circle $\{\omega\in\mathbb{C}, |\omega| = 1\}$ is the image of the segment $\{0\leqslant\xi\leqslant 2\pi,\ \eta = 0\}$. The superposition of these functions $f(\zeta) = F(e^{-i\zeta})$ defines the conformal mapping on the half-strip Π onto the exterior of G :

$$\Pi\to\mathbb{R}^2\setminus G\ (\zeta = \xi + i\eta\to z = f(\zeta)). \tag{2.6}$$

Under this mapping the image of the segment $\{0\leqslant\xi\leqslant 2\pi, \eta = 0\}$ is ∂G and $f(\pi) = z_0$.

The transformation (2.6) in the explict form can be obtained only for very special domains G. For the numerical calculations of the flow in an arbitrary domain the numerical approximations of these transorms can be used. In our calcuations [2] and [3] we used the method of K.I.Babenko (see [8], p. 374–379).

2.3 Change of variables in the problem (2.1)–(2.4). Let $z = x(\xi,\eta) + iy(\xi,\eta) = f(\xi + i\eta)$ be the transformation (2.6). We keep the notation φ for the potential in the new variables (ξ,η):

$$\varphi(\xi,\eta) = \varphi(\mathrm{Re} f\ (\xi + i\eta), \mathrm{Im} f\ (\xi + i\eta)).$$

For given γ we define the onflow potential with this circulation:

$$\varphi_\infty(\xi,\eta,\gamma) = x(\xi,\eta)\cos\alpha + y(\xi,\eta)\sin\alpha + \gamma\xi \tag{2.7}$$

and pose

$$\varphi(\xi,\eta) = \varphi_\infty(\xi,\eta,\gamma) - \Phi(\xi,\eta,\gamma). \tag{2.8}$$

As the transformation of variables $(\xi,\eta)\to(x,y)$ is defined by means of harmonic functions $x(\xi,\eta)$, $y(\xi,\eta)$ and onflow potential (2.7) is also a harmonic function, equation (2.1) in these variables takes form

$$\frac{\partial}{\partial\xi}\Big(R(\Phi)\frac{\partial(\Phi-\varphi_\infty)}{\partial\xi}\Big) + \frac{\partial}{\partial\eta}\Big(R(\Phi)\frac{\partial(\Phi-\varphi_\infty)}{\partial\eta}\Big) = 0, \tag{2.9}$$

where

$$R(\Phi) = \left(1 + \frac{k-1}{2} M_\infty^2 (1 - |\nabla_{(\xi,\eta)}(\Phi - \varphi_\infty)|^2 g^{-2}\right)^{\frac{1}{k-1}}, \tag{2.9'}$$

and

$$g(\xi, \eta) = |f'(\xi + i\eta)|$$

is the Jacobian of our transformation. Nonpenetration condition (2.2) is reduced to the Neumann's condition for $\eta = 0$:

$$\left.\frac{\partial \Phi}{\partial \eta}\right|_{\eta=0} = x_\eta(\xi, 0)\cos\alpha + y_\eta(\xi, 0)\sin\alpha. \tag{2.10}$$

To this condition we must add the periodicity condition and the condition of decay of Φ for $\eta \to \infty$,

$$\Phi(0, \eta) = \Phi(2\pi, \eta), \quad \Phi \in H^1(\Pi). \tag{2.11}$$

Now let us discuss the Kutta-Zhukovskii (K-Z) condition. We have $|\nabla_{x,y}\varphi| = |\nabla_{(\xi,\eta)}\varphi|(x_\xi^2 + y_\xi^2)^{-\frac{1}{2}}$. As the function x_ξ, y_ξ, x_η, y_η are equal to zero at the preimage $(\pi, 0)$ of the trailing edge z_0, we have that K-Z condition is satisfied if and only if

$$(\varphi_\xi^2 + \varphi_\eta^2)(\pi, 0) = 0.$$

As $\varphi_\eta(\pi, 0) = 0$ according to the nonpenetration condition (2.10), we come to the final form of the Kutta-Zhukovskii condition

$$\Phi_\xi(\pi, 0) - \gamma = 0 \tag{2.12}$$

Resume. The conformal mapping (2.6) permits us to reduce the problem (2.1)–(2.4) to the following boundary value problem: find function $\Phi(\xi, \eta, \gamma)$ and circulation γ from equations (2.9) and (2.9′), the boundary conditions (2.10), (2.11) and the additional condition (2.12).

2.4 Integral form of the problem (2.9)–(2.13). Multiplying equation (2.8) by a test function and integrating by parts we reduce the problem (2.9)– (2.13) to the follouing one: find the periodic with respect to ξ function Φ and the number γ from the following relations:

$$F(\Phi, \gamma, \omega) = 0, \quad \forall \omega \in H^1(\Pi), \tag{2.13}$$

$$\Phi_\xi(\pi, 0) - \gamma = 0, \tag{2.14}$$

where

$$F(\Phi, \gamma, \omega) = \iint_\Pi \left(R(\Phi)\nabla(\Phi - x\cos\alpha - y\sin\alpha - \gamma\xi) + \right.$$
$$\left. \nabla(x\cos\alpha + y\sin\alpha + \gamma\xi), \nabla\omega\right) d\xi d\eta + \int_0^{2\pi} \left((x_\eta \cos\alpha + y_\eta \sin\alpha)\omega\right)(\xi, 0) d\xi. \tag{2.15}$$

2.5 Linearization of the problem (2.13)–(2.15). Suppose that $(\Phi, \gamma) = (\Phi^{(n)}, \gamma^{(n)})$ is the n-iteration of the solution of the problem (2.13)–(2.15). We seek the correction $(\Phi^{(n+1)}, \gamma^{(n+1)}) = (\Phi + \Psi, \gamma + \delta)$ of this iteration by Newton's method:

$$F(\Phi, \gamma, \Psi, \delta, \omega) = \left.\frac{d}{d\varepsilon} F(\Phi + \varepsilon\Psi, \gamma + \varepsilon\delta, \omega)\right|_{\varepsilon=0} = 0, \tag{2.16}$$

$$\Psi_\xi(\pi, 0) - \delta = 0. \tag{2.17}$$

The direct calculation shows that the left-hand side of (2.16′) is equal to

$$\iint_{\Pi} \left[R(\Phi,\gamma)\langle\nabla\Psi,\nabla\omega\rangle - M_\infty^2 g^{-2} R^{2-k}(\Phi,\gamma)\langle\nabla\Psi,\nabla(\Phi-\varphi_\infty)\rangle \times \right.$$

$$\left.\langle\nabla\omega,\nabla(\Phi-\varphi_\infty)\rangle\right] d\xi d\eta + \delta \iint_{\Pi} \left[-R(\Phi,\gamma)\omega_\xi + \omega_\xi - \right. \tag{2.18}$$

$$\left. M_\infty^2 g^{-2} R^{2-k}(\Phi,\gamma)(\Phi-\varphi_\infty)_\xi \langle\nabla(\Phi-\varphi_\infty),\nabla\omega\rangle\right] d\xi d\eta.$$

From (2.18) it follows that (2.16) is a week formulation of the following problem (depending on Φ and γ):

$$\mathcal{R}(\Phi,\gamma)\Psi + H(\Phi,\gamma)\delta = -F(\Phi,\gamma). \tag{2.16′}$$

Here $\mathcal{R}$ is a second order partial differential operator defined on functions $\Psi \in H^1(\Pi)$ periodic with respect to ξ and satisfying zero Neumann's condition for $\eta = 0$, the first term in the (2.18) is the integral form of this operator. Equation (2.17) can be rewritten in the form

$$L(\Psi) - \delta = 0 \tag{2.17′}$$

where L is a linear functional acting on Ψ.

Suppose that the operator $\mathcal{R}$ has the inverse $\mathcal{R}^{-1}$. Applying this operator to (2.16′) we obtain

$$\Psi = -\mathcal{R}^{-1}(\Phi,\gamma)F(\Phi,\gamma) - \delta\mathcal{R}^{-1}(\Phi,\gamma)H(\Phi,\gamma).$$

Substituting the last expression in (2.17′) we can evaluate δ:

$$\delta = \frac{-L(\mathcal{R}^{-1}F)}{1 + L(\mathcal{R}^{-1}H)}. \tag{2.19}$$

Resume. The mechanical problem of the potential flow past a profile with an edge is reduced to the sequence of linear boundary value problems. On each step of the iterative process to obtain the potential and the circulation we twice solve the same boundary value problem with two different right-hand sides.

2.7 Zhukovskii formula. As a rule, the main goal of solving the flow problem is to find the lift and the drug, i.e.

$$R_y = \int_{\partial G} P\,dx, \quad R_x = \int_{\partial G} P\,dy \tag{2.20}$$

where the pressure P is given by the relation

$$P = \frac{1}{M_\infty^2 k}\left[1 + \frac{k-1}{2}M_\infty^2(1 - |\nabla\varphi|^2)\right]^{\frac{k}{k-1}}. \tag{2.21}$$

The famous Zhukovskii formulas which connect the lift, the drug and the circulation γ have the form

$$R_y = -\Gamma\cos\alpha, \quad R_x = \Gamma\sin\alpha, \quad \Gamma = 2\pi\gamma. \tag{2.22}$$

Remark. In numerical analysis of the problem we obtain the approximate value of the circulation and approximation of the potential φ. The relation (2.21) permits us to calculate the pressure and its integral along the profile ∂G. In other words, we can independently calculate the left and right-hand sides of (2.22). The differences of these values provide important characteristics of the numerical solution.

§3. Potential flows in double connected domains.

3.1 On (x, y) plane we consider a potential flow in the exterior of two bounded domains G_1 and G_2 (wing sections). We assume that the wings and, hence, these wing sections are smooth except for one corner at the trailing edge of G_1 and G_2 at the points z_1 and z_2 respectively.

General mathematical formulation of the problem is the same as in the case of simply-connected domain, the only (formal) difference is connected with the fact, that the onflow potential depends (linearly) on the circulations around G_1 and G_2.

Remark. The problem of a flow around an airfoil in the presence of the earth is trivially reduced to the above problem. In this case we seek the solution of the potential equation in the exterior of the domain G in the half-plane $y \geqslant 0$.

On the boundary ∂G and the line $y = 0$ the nonpenetration condition must be fulfilled. By symmetry the solution of this problem can be obtained from the solution corresponding to the exterrior of the bounded domains G and G', where G' is symmetric to G with respect to the x-axis, in this case the onflow is parallel to x-axis.

3.2 According to the general theory of holomorphic functions for every double-connected domain on $z = x + iy$ plane there exists a real number $r < 1$ and a univalent holomorphic function $F(\omega)$ which gives conformal mapping of the circular annulus $K = \{\omega \in \mathbb{C}, r \leqslant |\omega| \leqslant 1\}$ onto $\mathbb{C} \setminus (G_1 \cup G_2)$. The profiles ∂G_1 and ∂G_2 are the images of the circles $\{|\omega| = r\}$ and $\{|\omega| = 1\}$, respectively.

As the exterior of $G_1 \cup G_2$ contains infinity there exists a point ω_0 such that $F(\omega_0) = \infty$. We can suppose that ω_0 belongs to the real axis (in opposite case we perform the corresponding rotation of the annulus). The last condition defines uniquely the map F. Let us consider the superposition of F and the function $\omega = \exp(i\zeta)$ giving the map

$$M = \{\zeta_1 = \xi + i\eta,\ 0 \leqslant \xi \leqslant 2\pi,\ 0 \leqslant \eta \leqslant \omega\} \to K,\ \omega = \ln\frac{1}{r}.$$

Then we obtain the function $z = f(\zeta) = F(e^{i\zeta})$ which defines the conformal mapping of M onto $\mathbb{C} \setminus (G_1 \cup G_2)$. The segments $\{0 \leqslant \xi \leqslant 2\pi,\ \eta = 0\}$ and $\{0 \leqslant \xi \leqslant 2\pi,\ \eta = \omega\}$ are the preimages of ∂G_1 and ∂G_2, the points ζ_1 and ζ_2 are the preimages of the trailing edges z_1 and z_2, and ik, $2\pi + ik$, $0 < k < \omega$ are the preimages of the infinity.

The number r depends on the domain $\mathbb{C} \setminus (G_1 \cup G_2)$ and is called the conformal radius of the domain. This number is not known in advance and can be obtained simultaneously with the conformal mapping. In the numerical analysis of the flows the numerical approximation of the map f is used.

3.3 As in §2 we seek the solution of the flow problem in the form $\varphi = \varphi_\infty - \Phi$, where φ_∞ depends on the angle of attack α and circulations γ_1 and γ_2:

$$\varphi_\infty = x(\xi, \eta)\cos\alpha + y(\xi, \eta)\sin\alpha + \gamma_1\varphi_\infty^{(1)} + \gamma_2\varphi_\infty^{(2)},$$

where the circulation of $\varphi_\infty^{(j)}$ around a contour, containing ∂G_k equals to δ_k^j, j, k=1,2.

As $f(ik) = f(2\pi + ik) = \infty$, $\varphi_\infty(\xi, \eta) \to \infty$ when $\xi + i\eta \to ik$, $2\pi + ik$ and Φ defined on z-plane tends to zero at infinity, so it is reasonable to suppose that the function $\Phi(\xi, \eta)$ satisfies the following condition:

$$\Phi(0, k) = \Phi(2\pi, k) = 0 \tag{3.1}$$

Now we come to the following problem: find the triple $\{\Phi, \gamma_1, \gamma_2\}$ satisfying the equation

$$\frac{\partial}{\partial\xi}\left(\rho\frac{\partial(\Phi - \varphi_\infty)}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\rho\frac{\partial(\Phi - \varphi_\infty)}{\partial\eta}\right) = 0, \tag{3.2}$$

condition (3.1) and the following boundary conditions: $\Phi(0,\eta) = \Phi(2\pi,\eta)$,

$$\left.\frac{\partial(\Phi - \varphi_\infty)}{\partial \eta}\right|_{\eta=0,\omega} = 0, \quad \frac{\partial}{\partial \xi}(\Phi - \varphi_\infty)(\xi_j, 0) = 0 \; j = 1, 2.$$

If we seek the solution by means of Newton's method, we come to the sequence of the problems of the form

$$\mathcal{R}\Psi + \delta_1 H_1 + \delta_2 H_2 = F, \tag{3.3}$$

$$L_1(\Psi) + a_{11}\delta_1 + a_{12}\delta_2 = 0, \quad L_2(\Psi) + a_{21}\delta_1 + a_{22}\delta_2 = 0, \tag{3.4}$$

where $\mathcal{R}$ is the linear partial differential operator acting on functions Ψ in Π satisfying the periodicity condition with respect to ξ, the homogenous Neumann condition for $\eta = 0, \omega$ and the condition (3.1), and L_1 and L_2 are the continuous linear functionals. If we know the operator $\mathcal{R}^{-1}$, we can easily solve the last problem and find Ψ, δ_1 and δ_2.

References

1. A.I.Aptekarev, L.R.Volevich, *Application of finite elements of high order to numerical analysis of potential subsonic compressible flow*, Soviet Union–Japan Symposium on Computational Fluid Dynamics. Khabarovsk. 1988. Moscow Computer Center Acad. Sci. USSR, 1990.
2. A.I.Aptekarev, L.R.Volevich, *Finite elements method in the problem of a compressible potential subsonic flow.*, Preprint Keldysh Inst. Appl. Math. Acad. Sci. USSR **74 (in russian)** (1988).
3. A.I.Aptekarev, L.R.Volevich, *Computation of an axisymmetric potential flow around smooth profiles and profiles with n edge by means of finite elements of high order*, Preprint Keldysh Inst. Appl. Math. Acad. Sci. USSR **101 (in russian)** (1988).
4. A.I.Aptekarev, L.R.Volevich, *Computation of a subsonic potential flow around smooth profiles and profiles with an edge by means of finite elements of high order*, Preprint Keldysh Inst. Appl. Math. Acad. Sci. USSR **103 (in russian)** (1988).
5. A.I.Aptekarev, L.R.Volevich, *Computation of potential flows with non-zero angle of attack around axisimmetric bodies.*, Preprint Keldysh Inst. Appl. Math. Acad. Sci. USSR **5 (in russian)** (1992).
6. H.Berger, G.Warnecke, W.Wendland, *Finite elements for transonic potential flows*, Seminar analysis und andwendungen. (to appear in: Numer. Math. Partial Diff. Eq. (1988).
7. M.Shiffman, *On the existence of subsonic flows of a compressible fluid*, J. Rat. Anal. **1** (1952), 605–652.
8. K.I.Babenko, *Foundations of numerical analysis*, Moscow, Nauka, 1986 (in russian).
9. M.Hafes, J.South, E.Murman, *Artifical compressibility methods for numerical solutions of transonic full potential equation*, AIAA J.V. **17** (1979), no. 8, 838–844.